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ORIGINAL PAPER

Solutions to linear matrix ordinary differential equations via minimal, regular, and excessive space extension based universalization

Convergence and error estimates for truncation approximants in the homogeneous case with premultiplying polynomial coefficient matrix

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Abstract This work focuses on the solution of the linear matrix ordinary differential equations where the first derivative of the unknown matrix is equal to the same unknown matrix premultiplied by a given matrix polynomially varying with the independent variable. Work aims to get a universal form for this equation by using the space extension concept where new unknowns are defined to get more amenable form for the equation. The convergence of the series solution to this equation obtained via minimal, regular, and excessive space extension is also investigated with the aid of an appropriate norm analysis which also enables us to get error estimates for the truncated series solutions. A few illustrative examples are presented for practical convergence issues like approximation quality.

Keywords Ordinary differential equations · Space extension · Series solutions · Majorant series · Error estimates

1 Introduction

We focus on the following matrix ordinary differential equation (MODE)

$$\mathbf{X}'(t) = \mathbf{A}(t)\mathbf{X}(t), \tag{1}$$

where $\mathbf{X}(t)$ and $\mathbf{A}(t)$ stand for $n \times n$ type unknown and known matrix valued functions of the independent variable t, respectively. The independent variable t and the

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matrix functions $\mathbf{X}(t)$ and $\mathbf{A}(t)$ are all assumed to be real valued entities for simplicity. However, we will mention about their complex values when it be comes necessary to do so in our analysis especially in the norm analyses. Real analysis is sufficient for our purposes, unless we need to change the basic definitions of the linear vector space concepts such as distance, norm and inner product for certain complex analysis. We confine ourselves to the cases where $\mathbf{A}(t)$ is a polynomial of t. This is a specification in the more general class of matrix functions which are analytic everywhere in the finite regions of the independent variable's complex plane since the functions of this wider class can be considered as the limiting forms of the polynomials under certain convergence conditions. Hence, the polynomiality, in one sense, brings the analyticity [1]. Thus we write

$$\mathbf{A}(t) \equiv \sum_{i=0}^{m} t^i \mathbf{A}_i, \tag{2}$$

where the indexed $\mathbf{A}$s are t–independent $n \times n$ type given matrices [2]. This structure of the $\mathbf{A}(t)$ matrix function makes the MODE given in (1) regular everywhere in the finite regions of the complex plane of the independent variable t. In other words, (1) has no singularity any where in the finite regions of the t-complex plane. This means that the solution of (1) should have Taylor series expansions at finite points of the t-complex plane [3]. Hence we can write the following Maclaurin expansion

$$\mathbf{X}(t) = \sum_{i=0}^{\infty} t^i \mathbf{X}_i, \tag{3}$$

where $\mathbf{X}_i$s are t independent $n \times n$ type matrices. The use of this relation in (1) produces the following recursion

$$i\, \mathbf{X}_i = \sum_{j=0}^{i-1} \mathbf{A}_j \mathbf{X}_{i-j-1}, \qquad i = 0, 1, 2, \ldots \tag{4}$$

where the finite sum vanishes when its upper limit becomes less than the lower limit by leaving $\mathbf{X}_0$ undetermined. This arbitrariness can be removed by imposing an initial condition to (1) as follows

$$\mathbf{X}(0) = \mathbf{I}_n, \tag{5}$$

where $\mathbf{I}_n$ stands for the $n \times n$ type unit matrix. This choice does not cause any loss of generality since it is always possible to obtain a solution for a different initial matrix by postmultiplying this specific solution with that initial matrix. The reflection of this condition to the recursion in (4) is the following initialization

$$\mathbf{X}_0 = \mathbf{I}_n, \tag{6}$$

whose utilization in (4) produces a unique solution. The resulting series converges in a disk centered at $t = 0$ with a radius equal to the distance from $t = 0$ to the closest singularity of the solution in t-complex plane. The singularity in the solution is tightly related to the singularities of the matrix valued coefficient function $\mathbf{A}(t)$. Since we use polynomial structure in this work, the matrix $\mathbf{A}(t)$ has no singularity at any finite point of the complex plane of the t variable. Although the details of this relation can be given through an appropriate norm analysis we do not intend to do so here because we do not want to use this expansion straightforwardly. This underlies the fact that the recursion in (4) is quite cumbersome to be used for analytical purposes due to its infinite order, or in other words, nonlocal nature (the difference between the highest and lowest index values is not constant, it increases as i grows) despite its linearity. We will try to construct another MODE whose series solution has just a two term recursion, in other words, first order differen ce equation. We will use the space extension concept to this end. The convergence and error analysis issues will also be included in this paper.

As we will show later the resulting universal MODE has singularities at $t = 0$ and the series expansion's numerical convergence, or in other words, truncation approximation quality tends to decrease as t goes away from 0. This urges us to use some other series expansions at some other points sufficiently far from 0. However, this approach takes us to not a two term but three term recursion with index depending coefficients. The analytical solution of that recursion may not be easily and efficiently obtained unless very specific circumstances are encountered. Hence, we do not use the series expansion at another point sufficiently far from 0. Instead we construct a perturbation expansion which is rapidly con verging and has a two term recursion. This issue of perturbation expansion will be separately presented in a separate paper soon.

The present paper, which is devoted to the investigation of the series solution at $t = 0$, is organized as follows. The Sect. 2 involves the universalization via space extension. The Sect. 3 covers the convergence analysis of the whole series while the Sect. 4 is designed to give error analysis for the truncation approximations. The Sect. 5 focuses on the space extension concept again, however this time, by emphasizing on its classification via introduction the excessive space extension possibility. The Sect. 6 includes certain illustrative implementations while the Sect. 7 finalizes the paper with concluding re marks together with some comments for further readings.

2 Universalization via space extension

As we stated in the previous section the solution of (1) can be expressed as a Maclaurin series in t. It is always possible to separate a power series to some subseries by picking the powers which can be expressed as a multiple of a common positive integer plus a common remainder such that remainder varies from subseries to subseries while the common integer remains same. For example, a power series can always be expressed as a sum of an even and odd series (even means even powers are contained only, odd should be interpreted as involving the odd powers only [4]). Now we can write

$$\mathbf{X}(t) \equiv \sum_{i=0}^{m} t^i \mathbf{Y}_i(t), \tag{7}$$

where $n \times n$ type matrix $\mathbf{Y}_i(t)$ is assumed to be a series in powers of not t but t^{m+1}. This enables us to write

$$\mathbf{X}'(t) \equiv \sum_{i=0}^{m} i t^{i-1} \mathbf{Y}_i(t) + \sum_{i=0}^{m} t^i \mathbf{Y}_i'(t) \equiv \sum_{i=0}^{m} t^{i-1} \left[i\mathbf{Y}_i(t) + t\mathbf{Y}_i'(t) \right], \tag{8}$$

where the function between the left and right brackets are expressible in powers of t^{m+1}. We can continue and consider the following equality where the intermediate construction details are skipped

$$\left(\sum_{i=0}^{m} t^i \mathbf{A}_i \right) \left(\sum_{i=0}^{m} t^i \mathbf{Y}_i \right) = \sum_{i=0}^{m} t^i \left[\sum_{j=0}^{i} \mathbf{A}_j \mathbf{Y}_{i-j} + t^{m+1} \sum_{j=i+1}^{m} \mathbf{A}_j \mathbf{Y}_{i+m+1-j} \right]. \tag{9}$$

The second inner sum at the right hand side is assumed to be vanishing when i becomes m (this is in harmony with the convention that a sum whose upper bound is smaller than its lower bound should be taken zero). (1) can be obtained by equating the left hand sides of (8) and (9). The right hand side of (8) is composed of terms each of which is multiplied by a separate power of t between -1 and m inclusive and beyond these factors all terms are expressible in power series of t^{m+1}. Hence these powers make the corresponding terms linearly dependent or independent. There are $m + 2$ separate powers appearing in these right hand sides despite the number of unknown matrices is $m+1$. This excess power is due to the fact that t^m and t^{-1} containing terms are not completely independent in these formulae, since the factor t^m can be written as the product of t^{m+1} and t^{-1}. Hence it is better to choose to use either t^m or t^{-1} alone. We choose t^{-1} here and therefore rewrite (9) as follows by separating the term corresponding to $i = m$ from the finite sum and then expressing it in terms of t^{-1}

$$\begin{aligned} \left(\sum_{i=0}^{m} t^i \mathbf{A}_i \right) \left(\sum_{i=0}^{m} t^i \mathbf{Y}_i \right) &= t^{-1} \sum_{j=0}^{m} t^{m+1} \mathbf{A}_j \mathbf{Y}_{m-j} \\ &\quad + \sum_{i=0}^{m-1} t^i \left[\sum_{j=0}^{i} \mathbf{A}_j \mathbf{Y}_{i-j} + t^{m+1} \sum_{j=i+1}^{m} \mathbf{A}_j \mathbf{Y}_{i+m+1-j} \right]. \end{aligned} \tag{10}$$

The combination of (10) with (8) produces the following equation

$$\begin{aligned} &\sum_{i=0}^{m} t^{i-1} \left[i\mathbf{Y}_i(t) + t\mathbf{Y}_i'(t) \right] - t^{-1} \sum_{j=0}^{m} t^{m+1} \mathbf{A}_j \mathbf{Y}_{m-j} \\ &\quad - \sum_{i=0}^{m-1} t^i \left[\sum_{j=0}^{i} \mathbf{A}_j \mathbf{Y}_{i-j} + t^{m+1} \sum_{j=i+1}^{m} \mathbf{A}_j \mathbf{Y}_{i+m+1-j} \right] = \mathbf{0}. \end{aligned} \tag{11}$$

By changing the rightmost sum index from i to $i-1$ and using common finite sums we obtain

$$t^{-1}\left[t\mathbf{Y}_0'(t)-\sum_{j=0}^{m}t^{m+1}\mathbf{A}_j\mathbf{Y}_{m-j}\right] +\sum_{i=1}^{m}t^{i-1}\left[i\mathbf{Y}_i(t)+t\mathbf{Y}_i'(t)-\sum_{j=0}^{i-1}\mathbf{A}_j\mathbf{Y}_{i-j-1}-t^{m+1}\sum_{j=i}^{m}\mathbf{A}_j\mathbf{Y}_{i+m-j}\right]=\mathbf{0}, \tag{12}$$

where there are totally $m+1$ terms containing the factors t^{-1}, t^0, …, and, t^{m-1}. Since these terms are linearly independent, they should vanish individually. This gives

$$\begin{aligned} t\mathbf{Y}_0'(t)-\sum_{j=0}^{m}t^{m+1}\mathbf{A}_j\mathbf{Y}_{m-j}&=\mathbf{0},\\ i\mathbf{Y}_i(t)+t\mathbf{Y}_i'(t)-\sum_{j=0}^{i-1}\mathbf{A}_j\mathbf{Y}_{i-j-1}-t^{m+1}\sum_{j=i}^{m}\mathbf{A}_j\mathbf{Y}_{i+m-j}&=\mathbf{0}, \end{aligned} \tag{13}$$

which can be reorganized to be rewritten in the following form

$$\mathbf{Y}_i'(t)=-\frac{i}{t}\mathbf{Y}_i(t)+\frac{1}{t}\sum_{j=0}^{i-1}\mathbf{A}_j\mathbf{Y}_{i-j-1}+t^m\sum_{j=i}^{m}\mathbf{A}_j\mathbf{Y}_{i+m-j} \quad i=0,1,\ldots,m. \tag{14}$$

The vanishing convention for the sums whose upper bounds are less than their lower bounds is used here. The equations in (14) can be combined into a single equation as follows

$$\mathbf{Z}'(t)=\left[\frac{1}{t}\mathbf{B}_1+t^m\mathbf{B}_2\right]\mathbf{Z}(t), \tag{15}$$

where

$$\mathbf{Z}(t)\equiv\left[\ \mathbf{Y}_0(t)^T\ \ldots\ \mathbf{Y}_m(t)^T\ \right]^T, \tag{16}$$

$$\mathbf{B}_1\equiv\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0}\\ \mathbf{A}_0 & -\mathbf{I}_n & \mathbf{0} & \cdots & \mathbf{0}\\ \mathbf{A}_1 & \mathbf{A}_0 & -2\mathbf{I}_n & \ddots & \mathbf{0}\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ \mathbf{A}_{m-1} & \cdots & \mathbf{A}_1 & \mathbf{A}_0 & -m\mathbf{I}_n \end{bmatrix}, \tag{17}$$

$$\mathbf{B}_2 \equiv \begin{bmatrix} \mathbf{A}_m & \mathbf{A}_{m-1} & \cdots & \cdots & \mathbf{A}_0 \\ \mathbf{0} & \mathbf{A}_m & \mathbf{A}_{m-1} & \cdots & \mathbf{A}_1 \\ \mathbf{0} & \mathbf{0} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} & \mathbf{A}_m \end{bmatrix}. \tag{18}$$

We can propose the following series expansion as a solution to (15)

$$\mathbf{Z}(t) \equiv \sum_{i=0}^{\infty} t^{(m+1)i+\alpha}\mathbf{Z}_i, \tag{19}$$

where the unknown scalar exponent α is used because of the singularity of (15) at $t = 0$ [4]. The unknown $\mathbf{Z}_i$ coefficients stand for $(m+1)n \times n$ matrices which are independent of t. The utilization of this series expansion in (15) takes us to the following two term recursion

$$[((m+1)i+\alpha)\mathbf{I} - \mathbf{B}_1]\mathbf{Z}_i = \mathbf{B}_2\mathbf{Z}_{i-1}, \qquad i = 0, 1, 2, \ldots \tag{20}$$

where we take $\mathbf{Z}_{-1}$ identically vanishing. The initial form of this recursion gives the spectral problem of the matrix $\mathbf{B}_1$ as follows

$$[\alpha\mathbf{I} - \mathbf{B}_1]\mathbf{Z}_0 = \mathbf{0}, \tag{21}$$

which implies that the possible values for the exponent α are the eigenvalues of the matrix $\mathbf{B}_1$ [3]. Since $\mathbf{B}_1$ is a block triangular matrix with diagonal blocks proportional to the unit matrix $\mathbf{I}_n$ its eigenvalues are these proportionality coefficients which are $0, -1, \ldots, m-1$ with a common multiplicity of n. All eigenvalues' algebraic and geometric multiplicities are same hence there is a sufficient number of linearly independent eigenvectors to each eigenvalue [5]. This linear independency removes one possibility of the logarithmic solutions [6]. The other possibility which appears when some eigenvalues follow each others by integers to make the recursion unsolvable at some index values is not effective here because the differences between the eigenvalues can not be multiple of $m+1$. Thus, the above recursion has an invertible matrix coefficient at the left hand side for all positive integer i values. This enables us to get the following new explicit form of the recursion

$$\mathbf{Z}_i = [((m+1)i+\alpha)\mathbf{I} - \mathbf{B}_1]^{-1}\mathbf{B}_2\mathbf{Z}_{i-1}, \qquad i = 1, 2, \ldots \tag{22}$$

whose explicit formal solution can be obtained as follows

$$\mathbf{Z}_i = \left[\prod_{j=0}^{i-1}[((m+1)(i-j)+\alpha)\mathbf{I} - \mathbf{B}_1]^{-1}\mathbf{B}_2\right]\mathbf{Z}_0, \qquad i = 1, 2, \ldots \tag{23}$$

This result shows that the solution above can be unique only when $\mathbf{Z}_0$ is somehow determined uniquely. To this end we can explicitly write first

$$\mathbf{Z}_0 = \left[\mathbf{Z}_{0,0}^T \; \ldots \; \mathbf{Z}_{0,m}^T\right]^T, \tag{24}$$

and then consider (21). If α is chosen as an integer between $-(m-1)$ and 0 inclusive say m_1 then we can show that all $\mathbf{Z}_{0,j}$ matrices for j values less than m_1 vanish while $\mathbf{Z}_{0,m_1}$ remains arbitrary. This arbitrary matrix say $\mathbf{C}$ appears as the rightmost factor of all other nonzero components. In other words, arbitrariness is at the level of $n \times n$ matrices. For example, in the case where $\alpha = 0$ we can write

$$\mathbf{Z}_{0,0} = \mathbf{C}, \qquad \mathbf{Z}_{0,1} = \mathbf{A}_0\mathbf{C}, \qquad \mathbf{Z}_{0,2} = \frac{1}{2}\left(\mathbf{A}_0^2 + \mathbf{A}_1\right)\mathbf{C}, \tag{25}$$

or more generally in recursive form

$$i\,\mathbf{Z}_{0,i} = \sum_{j=0}^{i-1} \mathbf{A}_{i-j-1}\mathbf{Z}_{0,j}, \qquad i = 0, 1, \ldots, m, \tag{26}$$

where we have used the previous vanishing convention for the sums with smaller upper bounds than the lower ones. The recursion in (26) matches with the first $m+1$ equations of the recursion amongst the matrix coefficients of the series solution to (1). This means that if $\mathbf{Z}_{0,0}$ matches $\mathbf{I}_n$ which was the initial condition to (1) then the $n \times n$ block elements of the matrix $\mathbf{Z}_0$ match with the first corresponding Maclaurin expansion coefficients of the solution to (1). This implies that the matrix function $\mathbf{T}(t)^T\mathbf{Z}(t)$ for $\alpha = 0$ where

$$\mathbf{T}(t) \equiv \left[\,\mathbf{I} \;\; t\mathbf{I} \;\; \ldots \;\; t^m\mathbf{I}\,\right]^T \tag{27}$$

is the solution of (1) under the unit matrix initial condition. All these mean that it is sufficient to consider the case where $\alpha = 0$ for the universal MODE given through (15). This is valid under the condition that the initial matrix $\mathbf{Z}_0$'s components satisfy (26) together with $\mathbf{Z}_{0,0} = \mathbf{I}_n$. We call the MODE in (26) "Universalized Form". The matrix $\mathbf{Z}(t)$ is of $(m+1)n \times n$ type while the matrix unknown $\mathbf{X}(t)$ in (1) is an $n \times n$ array. This means that the space over (26) is composed of matrices having $(m+1)$ times more elements than the matrices of the space constructed over (1). Hence we may call this universalization procedure "Space Extension" [7,8]. The benefit of the space extension is the reduction of the recursions coming from the series solution to two term including ones at the expense of an increment in the number of the unknown entities.

3 Convergence issues

The MODE given in (15) has its only singularity where the solution remains finite at $t = 0$. Hence one can expect that the series expansions at $t = 0$ converge in a disk centered at the origin of the t-complex plane. However it is better to prove this convergence since the formulation may give certain information about the approximation quality. To this end, we start from (19) by taking the norm of its both side and then use the fact that the norm of a series is less than or equal to the same type series over the norms of its elements [9]. We can write

$$\|\mathbf{Z}(t)\| \leq \sum_{i=0}^{\infty} |t|^{(m+1)i+\alpha} \|\mathbf{Z}_i\|, \tag{28}$$

where the norm we have used is over the matrices. It does not involve the functions. Hence t appears as a parameter here. The norm above is assumed to be submultiplicative. In other words, the norm of a product is less than or equal to the product of the factor norms. This property enables us to write the following inequality from (22)

$$\|\mathbf{Z}_i\| \leq \left\| [((m+1)i+\alpha)\mathbf{I} - \mathbf{B}_1]^{-1} \mathbf{B}_2 \right\| \|\mathbf{Z}_{i-1}\|, \qquad i = 1, 2, \ldots \tag{29}$$

Our purpose is to find a rather easy form of bound to the first norm at the right hand side of this inequality. For this reason, we define

$$\mathbf{L} \equiv [((m+1)i+\alpha)\mathbf{I} - \mathbf{B}_1] \equiv \mathbf{L}_D - \mathbf{L}_O, \tag{30}$$

where $\mathbf{L}_D$ and $-\mathbf{L}_O$ stand for the block diagonal and off diagonal parts, respectively. Since we have used the minus sign above $\mathbf{L}_O$ is the lower triangular part of the matrix $\mathbf{B}_1$ with zero diagonal blocks. Hence $\mathbf{L}_D$ is composed of $m+1$ block at its main diagonal such that its jth component is $((m+1)i+\alpha+j-1)\mathbf{I}$ where $j = 1, 2, \ldots, m+1$. This permits us to write

$$\begin{aligned}
[((m+1)i+\alpha)\mathbf{I} - \mathbf{B}_1]^{-1} \mathbf{B}_2 &= [\mathbf{L}_D - \mathbf{L}_O]^{-1} \mathbf{B}_2 = \left[\mathbf{I} - \mathbf{L}_D^{-1}\mathbf{L}_O\right]^{-1} \mathbf{L}_D^{-1}\mathbf{B}_2 \\
&= \sum_{k=0}^{\infty} \left[\mathbf{L}_D^{-1}\mathbf{L}_O\right]^k \mathbf{L}_D^{-1}\mathbf{B}_2 = \sum_{k=0}^{m} \left[\mathbf{L}_D^{-1}\mathbf{L}_O\right]^k \mathbf{L}_D^{-1}\mathbf{B}_2,
\end{aligned} \tag{31}$$

where we have use the fact that $\mathbf{L}_D^{-1}\mathbf{L}_O$ is lower block triangular with zero main diagonal blocks and therefore its all powers greater than m identically vanish. The last expression at the right hand side of above formula urges us to write [10]

$$\begin{aligned}
\left\| \sum_{k=0}^{m} \left[\mathbf{L}_D^{-1}\mathbf{L}_O\right]^k \mathbf{L}_D^{-1}\mathbf{B}_2 \right\| &\leq \sum_{k=0}^{m} \left\|\mathbf{L}_D^{-1}\mathbf{L}_O\right\|^k \left\|\mathbf{L}_D^{-1}\mathbf{B}_2\right\| \\
&= \frac{1 - \left\|\mathbf{L}_D^{-1}\mathbf{L}_O\right\|^{m+1}}{1 - \left\|\mathbf{L}_D^{-1}\mathbf{L}_O\right\|} \left\|\mathbf{L}_D^{-1}\mathbf{B}_2\right\|,
\end{aligned} \tag{32}$$

which makes it possible to write the following inequality from (31)

$$\left\| \left[((m+1)i+\alpha)\mathbf{I}-\mathbf{B}_1 \right]^{-1}\mathbf{B}_2 \right\| \leq \frac{1-\left\| \mathbf{L}_D^{-1}\mathbf{L}_O \right\|^{m+1}}{1-\left\| \mathbf{L}_D^{-1}\mathbf{L}_O \right\|} \left\| \mathbf{L}_D^{-1}\mathbf{B}_2 \right\|. \tag{33}$$

Although we have not specified until now, we prefer to use Frobenius norm for matrices because of its many pleasent properties facilitating the bound analysis [11]. Now, since $\mathbf{L}_D^{-1}$ multiplies the jth block row of its operand by the reciprocal of $(m+1)i+\alpha+j-1$ where $j=1,2,\ldots,m+1$, we can write the following inequalities by skipping the intermediate details

$$\left\| \mathbf{L}_D^{-1}\mathbf{L}_O \right\| \leq \frac{1}{(m+1)i+\alpha}\left\| \mathbf{L}_O \right\|, \qquad \left\| \mathbf{L}_D^{-1}\mathbf{B}_2 \right\| \leq \frac{1}{(m+1)i+\alpha}\left\| \mathbf{B}_2 \right\|, \tag{34}$$

where α can be taken zero since our ultimate evaluations involve the cases where α takes 0 value only. Although these results may be considered more pessimistic than they should be, they are still useful for bound construction. Indeed, one can find a lower bound, say i_{lb}, for i in the first part of (34) to make the right hand side bound smaller than 1. This requires

$$i_{lb} > \frac{\left\| \mathbf{L}_O \right\|}{m+1}. \tag{35}$$

We could find another lower bound for i to make the bound in the second part of (34) smaller than 1. However, the rational expression at the right hand side of (33) is more important than its right neighbour factor, since its standing structure is not appropriate to find easy bounds for the series solution to (15). The restriction on i values in (35) is not problematic because it is sufficient to divide the series under consideration for bound analysis to two parts one of which is finite and then to deal with the infinite part to investigate the convergence. Now the formulae in (34) together with (35) can be combined with (33) to give the following more pessimistic inequality

$$\left\| \left[((m+1)i+\alpha)\mathbf{I}-\mathbf{B}_1 \right]^{-1}\mathbf{B}_2 \right\| \leq \left[(m+1)i-\left\| \mathbf{L}_O \right\| \right]^{-1}\left\| \mathbf{B}_2 \right\|, \quad i > i_{lb} \tag{36}$$

which can be reflected to (29) as follows

$$\left\| \mathbf{Z}_i \right\| \leq \frac{\left\| \mathbf{B}_2 \right\|}{m+1}\frac{1}{i-\frac{\left\| \mathbf{L}_O \right\|}{m+1}}\left\| \mathbf{Z}_{i-1} \right\|, \qquad i=i_{lb}+1, i_{lb}+2, \ldots \tag{37}$$

This relation implies that the convergence radius of the series solution to (15) is infinite. This was what we have expected from the nonsingular nature of (1) or the singularities of (15) at zero and infinity.

4 Truncation approximants and error estimates

We can define now

$$\mathbf{X}_{T,i}(t) \equiv \mathbf{T}(t)^T \mathbf{Z}_{T,i}(t), \qquad i = 0, 1, 2, \ldots \tag{38}$$

where, by only considering the case where α vanishes,

$$\mathbf{Z}_{T,i}(t) \equiv \sum_{j=0}^{i} t^{(m+1)j} \mathbf{Z}_j, \qquad i = 0, 1, 2, \ldots \tag{39}$$

to approximate the solution of (1). We call $\mathbf{X}_{T,i}(t)$ and $\mathbf{Z}_{T,i}(t)$ "ith Truncation Approximation" for (1) and (15), respectively. Our purpose in this section is to find error estimates for these entities. To this end, we will basically focus on (15)'s solution and define "ith Error Estimate" as follows

$$\mathbf{E}_i(t) \equiv \mathbf{Z}(t) - \mathbf{Z}_{T,i}(t) \equiv \sum_{j=i+1}^{\infty} t^{(m+1)j} \mathbf{Z}_j, \qquad i = 0, 1, 2, \ldots \tag{40}$$

Although the evaluation of $\mathbf{E}_i(t)$ seems not to be hard it is easier to attempt to find not itself but a norm bound to it. The i values in the last equation is permitted to be any nonnegative integer. However it is easier to deal with i values greater than i_{lb} mentioned in the previous section because these values facilitate the norm bound analysis. The cases where i values are less than i_{lb} the error term can be partitioned to a finite and an infinite portion the latter of which involves i values greater than i_{lb} only. Hence we confine ourselves to the cases where $i > i_{lb}$ without any loss of generality. We also consider the real values of t only. Beyond that we deal with the t values in the interval $[0, t_{\max}]$ where the interval length can be arbitrarily chosen. We write the following inequality from (40)

$$\|\mathbf{E}_i(t)\| \le \sum_{j=i+1}^{\infty} t^{(m+1)j} \left\|\mathbf{Z}_j\right\|, \qquad i = i_{lb}, i_{lb}+1, \ldots \tag{41}$$

and the new more pessimistic form of (37) as follows

$$\left\|\mathbf{Z}_j\right\| \le \frac{\|\mathbf{B}_2\|}{m+1} \frac{1}{j-i} \left\|\mathbf{Z}_{j-1}\right\|, \quad j = i+1, i+2, \ldots, \quad i = i_{lb}, i_{lb}+1, \ldots \tag{42}$$

which can be iterated to get

$$\left\|\mathbf{Z}_j\right\| \le \frac{\|\mathbf{B}_2\|^{j-i-1}}{(m+1)^{j-i-1}} \frac{1}{(j-i)!} \|\mathbf{Z}_{i+1}\|, \quad j = i+1, i+2, \ldots, \quad i = i_{lb}, i_{lb}+1, \ldots \tag{43}$$

Insertion of this result into (41) and then the replacement of j by $j+i+1$ gives

$$\|\mathbf{E}_i(t)\| \leq \sum_{j=0}^{\infty} t^{(m+1)(j+i+1)} \frac{\|\mathbf{B}_2\|^j}{(m+1)^j} \frac{1}{(j+1)!} \|\mathbf{Z}_{i+1}\|$$
$$= (m+1)\frac{\|\mathbf{Z}_{i+1}\|}{\|\mathbf{B}_2\|} t^{(m+1)i} \left(\mathrm{e}^{\frac{\|\mathbf{B}_2\|}{m+1} t^{m+1}} - 1 \right), \quad i = i_{lb}, i_{lb}+1 \ldots \tag{44}$$

which implies that the truncation error increases as t and/or the norm of the matrix $\mathbf{B}_2$ grow. On the other hand, greater values of i result in smaller errors because of the factor $\|\mathbf{Z}_{i+1}\|$ since it behaves like the reciprocal of $i!$. Although the error estimate in (44) has a simple structure it is based on too relaxed inequalities. This may screen certain existing pleasent properties and one may seek more optimistic, in other words, much tight er error estimates. What we need to do so is to reconsider the inequality, obtained from (37) by rep lacing i by j, where the denominator of the second term at right has been deliberately too much diminished by replacing the minus term after j with $-i$. We could replace the same term with $-i_{lb}$ to get more optimism. If we do so we can write

$$\left\|\mathbf{Z}_j\right\| \leq \frac{\|\mathbf{B}_2\|}{m+1} \frac{1}{j-i_{lb}} \left\|\mathbf{Z}_{j-1}\right\|, \quad j = i+1, i+2, \ldots, \quad i = i_{lb}, i_{lb}+1, \ldots \tag{45}$$

which can be iterated to get

$$\left\|\mathbf{Z}_j\right\| \leq \frac{\|\mathbf{B}_2\|^{j-i-1}}{(m+1)^{j-i-1}} \frac{(i+1-i_{lb})!}{(j-i_{lb})!} \|\mathbf{Z}_{i+1}\|,$$
$$j = i+1, i+2, \ldots, \quad i = i_{lb}, i_{lb}+1, \ldots \tag{46}$$

Insertion of this result into (41) and then the replacement of j by $j+i+1$ gives

$$\|\mathbf{E}_i(t)\| \leq \sum_{j=0}^{\infty} t^{(m+1)(j+i+1)} \frac{\|\mathbf{B}_2\|^j}{(m+1)^j} \frac{(i+1-i_{lb})!}{(j+i+1-i_{lb})!} \|\mathbf{Z}_{i+1}\|$$
$$= \sum_{j=1}^{\infty} t^{(m+1)(j+i+1)} \frac{\|\mathbf{B}_2\|^j}{(m+1)^j} \frac{(i+1-i_{lb})!}{(j+i+1-i_{lb})!} \|\mathbf{Z}_{i+1}\|$$
$$+ t^{(m+1)(i+1)} \|\mathbf{Z}_{i+1}\|, \qquad i = i_{lb}, i_{lb}+1 \ldots \tag{47}$$

By using integral representation of beta function [12] it is possible to write

$$\frac{(i+1-i_{lb})!}{(i+1-i_{lb}+j)!} = \frac{1}{(j-1)!} \int_0^1 d\tau \tau^{j-1} (1-\tau)^{i+1-i_{lb}}, \tag{48}$$

which means

$$\sum_{j=1}^{\infty} t^{(m+1)(j+i+1)} \frac{\|\mathbf{B}_2\|^j}{(m+1)^j} \frac{(i+1-i_{lb})!}{(j+i+1-i_{lb})!} \|\mathbf{Z}_{i+1}\|$$
$$= \sum_{j=1}^{\infty} t^{(m+1)(j+i+1)} \frac{\|\mathbf{B}_2\|^j}{(m+1)^j} \frac{1}{(j-1)!} \int_0^1 d\tau \tau^{j-1} (1-\tau)^{i+1-i_{lb}} \|\mathbf{Z}_{i+1}\|$$
$$= \|\mathbf{Z}_{i+1}\| \frac{\|\mathbf{B}_2\|}{(m+1)} t^{(m+1)(i+2)} \int_0^1 d\tau (1-\tau)^{i+1-i_{lb}} \sum_{j=0}^{\infty} \frac{\|\mathbf{B}_2\|^j}{(m+1)^j} \frac{t^{(m+1)j}\tau^j}{j!}$$
$$= \|\mathbf{Z}_{i+1}\| \frac{\|\mathbf{B}_2\|}{(m+1)} t^{(m+1)(i+2)} \int_0^1 d\tau (1-\tau)^{i+1-i_{lb}} \mathrm{e}^{\frac{\|\mathbf{B}_2\|}{m+1} t^{m+1}\tau}. \tag{49}$$

The replacement of τ by τ^{m+1}/t^{m+1} in the rightmost integral of (49) enables us to write

$$\int_0^1 d\tau (1-\tau)^{i+1-i_{lb}} \mathrm{e}^{\frac{\|\mathbf{B}_2\|}{m+1} t^{m+1}\tau}$$
$$= (m+1) t^{-(m+1)(i+2-i_{lb})} \int_0^{t^{m+1}} d\tau \tau^m \left(t^{m+1} - \tau^{m+1}\right)^{i+1-i_{lb}} \mathrm{e}^{\frac{\|\mathbf{B}_2\|}{m+1} \tau^{m+1}}, \tag{50}$$

whose utilization in (49) gives

$$\sum_{j=1}^{\infty} t^{(m+1)(j+i+1)} \frac{\|\mathbf{B}_2\|^j}{(m+1)^j} \frac{(i+1-i_{lb})!}{(j+i+1-i_{lb})!} \|\mathbf{Z}_{i+1}\|$$
$$= \sum_{j=1}^{\infty} t^{(m+1)(j+i+1)} \frac{\|\mathbf{B}_2\|^j}{(m+1)^j} \frac{1}{(j-1)!} \int_0^1 d\tau \tau^{j-1} (1-\tau)^{i+1-i_{lb}} \|\mathbf{Z}_{i+1}\|$$
$$= \|\mathbf{Z}_{i+1}\| \|\mathbf{B}_2\| t^{(m+1)i_{lb}} \int_0^{t^{m+1}} d\tau \tau^m \left(t^{m+1} - \tau^{m+1}\right)^{i+1-i_{lb}} \mathrm{e}^{\frac{\|\mathbf{B}_2\|}{m+1} \tau^{m+1}}. \tag{51}$$

Finally from (49) we obtain

$$\|\mathbf{E}_i(t)\| \leq \|\mathbf{Z}_{i+1}\| \|\mathbf{B}_2\| t^{(m+1)i_{lb}} \int_0^{t^{m+1}} d\tau \tau^m \left(t^{m+1} - \tau^{m+1}\right)^{i+1-i_{lb}} \mathrm{e}^{\frac{\|\mathbf{B}_2\|}{m+1} \tau^{m+1}}$$
$$+ t^{(m+1)(i+1)} \|\mathbf{Z}_{i+1}\|, \qquad i = i_{lb}, i_{lb}+1 \ldots \tag{52}$$

This is an integral representation for error estimate. Its purely algebraic explicit counterpart can be obtained from the left side series of (49) through the following manipulations

$$\sum_{j=1}^{\infty} t^{(m+1)(j+i+1)} \frac{\|\mathbf{B}_2\|^j}{(m+1)^j} \frac{(i+1-i_{lb})!}{(j+i+1-i_{lb})!} \|\mathbf{Z}_{i+1}\|$$
$$= (i+1-i_{lb})! \, \|\mathbf{Z}_{i+1}\| \frac{(m+1)^{i+1-i_{lb}}}{\|\mathbf{B}_2\|^{i+1-i_{lb}}} t^{(m+1)i_{lb}} \sum_{i=i+2-i_{lb}}^{\infty} \frac{\|\mathbf{B}_2\|^j}{(m+1)^j} \frac{t^{(m+1)j}}{j!}$$
$$= (i+1-i_{lb})! \, \|\mathbf{Z}_{i+1}\| \frac{(m+1)^{i+1-i_{lb}}}{\|\mathbf{B}_2\|^{i+1-i_{lb}}} t^{(m+1)i_{lb}} \times \left(e^{\frac{\|\mathbf{B}_2\|}{m+1} t^{m+1}} - \sum_{j=0}^{i+1-i_{lb}} \frac{\|\mathbf{B}_2\|^j}{(m+1)^j} \frac{t^{(m+1)j}}{j!} \right). \tag{53}$$

If we define

$$\chi_i(t) \equiv \frac{(i+1)!}{t^{i+1}} \left(e^t - \sum_{j=0}^{i} \frac{t^j}{j!} \right), \qquad i = 0, 1, 2, \ldots \tag{54}$$

then (53) becomes

$$\sum_{j=1}^{\infty} t^{(m+1)(j+i+1)} \frac{\|\mathbf{B}_2\|^j}{(m+1)^j} \frac{(i+1-i_{lb})!}{(j+i+1-i_{lb})!} \|\mathbf{Z}_{i+1}\|$$
$$= \|\mathbf{Z}_{i+1}\| \frac{\|\mathbf{B}_2\|}{(m+1)} \frac{t^{(m+1)(i+2)}}{i+2-i_{lb}} \chi_{i+1-i_{lb}} \left(\frac{\|\mathbf{B}_2\|}{m+1} t^{m+1} \right), \tag{55}$$

whose combination with (47) gives

$$\|\mathbf{E}_i(t)\| \le \|\mathbf{Z}_{i+1}\| \frac{\|\mathbf{B}_2\|}{(m+1)} \frac{t^{(m+1)(i+2)}}{i+2-i_{lb}} \chi_{i+1-i_{lb}} \left(\frac{\|\mathbf{B}_2\|}{m+1} t^{m+1} \right) + t^{(m+1)(i+1)} \|\mathbf{Z}_{i+1}\|, \qquad i = i_{lb}, i_{lb}+1 \ldots \tag{56}$$

A careful look at the structure of the χ functions shows that these functions approach 1 as their arguments tend to vanish. This means that a prescribed accuracy can be achieved only when t remains less than a critical value which is reciprocally and directly proportional to the norm of the matrix $\mathbf{B}_2$ and $(m+1)$, respectively. These two entities are fixed at the beginning of the universalization. Hence every prescribed accuracy can only be obtained by working in an interval which is specific to that accuracy. As can be noticed from here, these estimates are tighter than the previously constructed one. We do not go beyond this point here since we find these bound analyses sufficient for our purposes.

5 Excessive space extension

We have chosen the space extension order as $m+1$ which is one more than the degree of the polynomial matrix co efficient $\mathbf{A}(t)$. This fixed choice removes the flexibility of magnitude reducing in the argument of χ function defined previously. However there is some how a tricky way to use space extension orders much high er than the degree of $\mathbf{A}(t)$. To this end we can consider $\mathbf{A}(t)$ as if an m_{ex} $(>m)$ degree polynomial of t and write

$$\mathbf{A}(t)=\sum_{i=0}^{m_{\text{ex}}} t^i \mathbf{A}_i, \tag{57}$$

where

$$\mathbf{A}_i \equiv \mathbf{0}, \qquad i=m+1,\ldots,m_{\text{ex}}. \tag{58}$$

The universalization of (1) via a space extension to $(m_{\text{ex}}+1)$th power of t produces

$$\mathbf{Z}'(t)=\left[\frac{1}{t}\mathbf{B}_1^{(\text{ex})}+t^{m_{\text{ex}}}\mathbf{B}_2^{(\text{ex})}\right]\mathbf{Z}(t), \tag{59}$$

where the matrices $\mathbf{B}_1^{(\text{ex})}$ and $t^{m_{\text{ex}}}\mathbf{B}_2^{(\text{ex})}$ are of $(m_{\text{ex}}+1)n \times (m_{\text{ex}}+1)$ type while the type of the unknown matrix $\mathbf{Z}(t)$ is $(m_{\text{ex}}+1)n \times n$. In other words the dimensions of the matrix entities increase. The matrices $\mathbf{B}_1^{(\text{ex})}$ and $t^{m_{\text{ex}}}\mathbf{B}_2^{(\text{ex})}$ are lower and upper triangular again. However this time $\mathbf{B}_1^{(\text{ex})}$ has zero lower diagonals parallel to main diagonal. Although the main diagonal does not vanish completely, its $m_{\text{ex}}-m$ lower neighbors are zero. A similar situation is encountered for $\mathbf{B}_2^{(\text{ex})}$. Its only nonzero block which is $\mathbf{B}_2$ is located at the rightuppermost position in a two by two block representation. These vanish ing natures facilitate a lot of things. It is easy to show that the Frobenius norms of $\mathbf{B}_2^{(\text{ex})}$ and $\mathbf{B}_2$ are identical. This enables m_{ex} to increase the numerical approximation quality. The error estimate of the previous section becomes diminishing as m_{ex} increases. The increasing sparsity in $\mathbf{B}_1^{(\text{ex})}$ also contributes to the numerical approximation quality positively. Hence, working with greater m_{ex} values increases the rapidity of numerical convergence.

6 Illustrative implementations

We consider first the case where $\mathbf{A}(t)$ is taken as follows

$$\mathbf{A}(t)\equiv\begin{bmatrix} -\frac{19}{2}t-12 & -14t-\frac{35}{2} \\ \frac{20}{3}t+\frac{25}{3} & \frac{59}{6}t+\frac{73}{6} \end{bmatrix}. \tag{60}$$

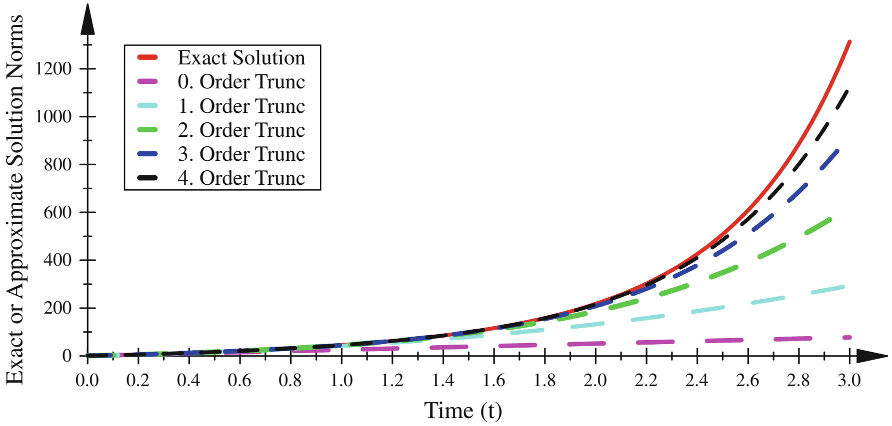

Fig. 1 The comparison of the Frobenius norms of the exact solution and first five truncation approximants constructed via minimal space extension for the system given through (60)

The analytic solution which becomes 2×2 type unit matrix at $t = 0$ can be found to (1) for this specific case. It is explicitly given below

$$\mathbf{X}(t) \equiv \begin{bmatrix} 15e^{-\frac{t^2}{12}-\frac{t}{3}} - 14e^{\frac{t^2}{4}+\frac{t}{2}} & 21e^{-\frac{t^2}{12}-\frac{t}{3}} - 21e^{\frac{t^2}{4}+\frac{t}{2}} \\ 10e^{\frac{t^2}{4}+\frac{t}{2}} - 10e^{-\frac{t^2}{12}-\frac{t}{3}} & 15e^{\frac{t^2}{4}+\frac{t}{2}} - 14e^{-\frac{t^2}{12}-\frac{t}{3}} \end{bmatrix}. \quad (61)$$

As a matter of fact this structure of $\mathbf{X}(t)$ is assumed to be a solution to (1) under the unit matrix initialization and then $\mathbf{A}(t)$ is determined from the equation. This $\mathbf{X}(t)$ is deliberately constructed by a similarity transformation from a 2×2 diagonal matrix whose elements are purely some exponential functions with second degree polynomial arguments to ensure the polynomial structure in $\mathbf{A}$. The explicit structures of the matrices $\mathbf{B}_1$, $\mathbf{B}_2$, and, $\mathbf{Z}_0$ are given below

$$\mathbf{B}_1 \equiv \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 12 & -35/2 & -1 & 0 \\ 25/3 & 73/6 & 0 & -1 \end{bmatrix}, \quad \mathbf{B}_2 \equiv \begin{bmatrix} -19/2 & -14 & -12 & -35/2 \\ 20/3 & 59/6 & 25/3 & 73/6 \\ 0 & 0 & -19/2 & -14 \\ 0 & 0 & 20/3 & 59/6 \end{bmatrix}, \quad (62)$$

$$\mathbf{Z}_0 \equiv \begin{bmatrix} 1 & 0 & -12 & 25/3 \\ 0 & 1 & -35/2 & 73/6 \end{bmatrix}^T. \quad (63)$$

First five truncation approximants constructed for mini mal space extension and the exact solution of the system are compared through the variation of their Frobenius norms in time over the interval [0, 3] in Fig. 1.

This figure is composed of six curves, uppermost solid one of which is for the exact solution's Frobenius norm while the other ones which are displayed via dashed curves with upward increasing thickness. They are also up ward ordered from the one term to five terms truncations. What we see from these plots is the fact that the increasing truncation order means better approximation quality. However there are all evidences for the quality loss as t gets apart from 0. Nevertheless, first five term seems to be representing the exact solution quite well in the interval [0, 3]. The curvature of the exponential functions in the previous example has been chosen rather small. This has been reflected a good quality of approximation even by using first five terms in a rather wide interval. However this may not be happening always in this way and our experiences (not only ours but someones's else) show that increasing curvature appears as a negative reflection on the quality. Hence, we choose our second example in almost same structure except the greater exponential factors in the functions to get higher curvature. The coefficient matrix is given as follows in this case

$$\mathbf{A}(t) \equiv \mathbf{A}_0 + t\mathbf{A}_1, \qquad \mathbf{A}_0 \equiv \begin{bmatrix} -117 & -168 \\ 80 & 115 \end{bmatrix}, \qquad \mathbf{A}_1 \equiv \begin{bmatrix} -202 & -294 \\ 140 & 204 \end{bmatrix}. \tag{64}$$

The analytic solution to (1), which becomes 2×2 type unit matrix at $t = 0$ can be explicitly given as follows

$$\mathbf{X}(t) \equiv \begin{bmatrix} 15e^{-3t^2-5t} - 14e^{4t^2+3t} & 21e^{-3t^2-5t} - 21e^{4t+3t} \\ 10e^{4t^2+3t} - 10e^{-3t^2-5t} & 15e^{4t^2+3t} - 14e^{-3t^2-5t} \end{bmatrix}. \tag{65}$$

The matrices $\mathbf{B}_1$ and $\mathbf{B}_2$ can be given through

$$\mathbf{B}_1 \equiv \begin{bmatrix} \mathbf{0}_2 & \mathbf{0}_2 \\ \mathbf{A}_0 & -\mathbf{I}_2 \end{bmatrix}, \qquad \mathbf{B}_2 \equiv \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_0 \\ \mathbf{0}_2 & -\mathbf{A}_1 \end{bmatrix}, \tag{66}$$

where $\mathbf{0}_2$ and $\mathbf{I}_2$ stand for 2×2 type zero and unit matrices, respectively. The initial matrix coefficient $\mathbf{Z}_0$ has the following explicit expression

$$\mathbf{Z}_0 \equiv \begin{bmatrix} \mathbf{I}_2 \\ \mathbf{A}_0 \end{bmatrix}. \tag{67}$$

First five truncation approximants constructed for minimal space extension and the exact solution of the system are compared through the variation of their Frobenius norms in time over the interval [0, 0.7] in Fig. 2.

The explanation is same as in Fig. 1. The behaviour of the approximants and the exact solution are also same in shape. However, the concerning interval of t is now narrower, [0, 0.7]. Apparently, the approximation qualities of the same truncations decrease in this case. What we have obtained for $t \in [0, 3]$ in the previous case is almost equivalent to the case for $t \in [0, 0.7]$.

Our third example is constructed for the same coefficient matrix $\mathbf{A}(t)$ given by (64). The only thing changed is the order of the space extension. In the previous cases we have concerned with the case where $m = 2$. It was the minimal space extension since the minimal order of extension has been chosen. In this case we use regular (minimal

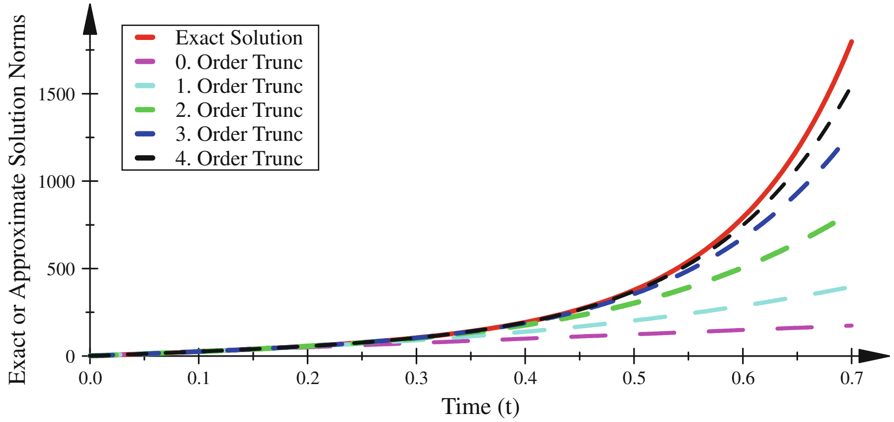

Fig. 2 The comparison of the Frobenius norms of the exact solution and first five truncation approximants constructed via minimal space extension for the system given through (64)

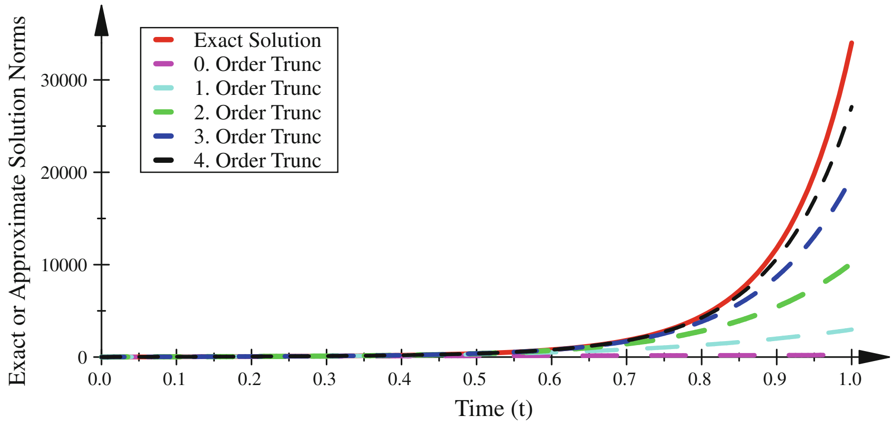

Fig. 3 The comparison of the Frobenius norms of the exact solution and first five truncation approximants constructed via regular (minimal excessive) space extension for the system given through (64)

excessive) space extension. Hence we take $m_{ex} = 2$. The matrices $\mathbf{B}_1$ and $\mathbf{B}_2$ together with the initial matrix $\mathbf{Z}_0$ are explicitly given below

$$\mathbf{B}_1 \equiv \begin{bmatrix} \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ \mathbf{A}_0 & -\mathbf{I}_2 & \mathbf{0}_2 \\ \mathbf{A}_1 & -\mathbf{A}_0 & -2\mathbf{I}_2 \end{bmatrix}, \qquad \mathbf{B}_2 \equiv \begin{bmatrix} \mathbf{0}_2 & \mathbf{A}_1 & \mathbf{A}_0 \\ \mathbf{0}_2 & \mathbf{0}_2 & -\mathbf{A}_1 \\ \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 \end{bmatrix}, \tag{68}$$

$$\mathbf{Z}_0 \equiv \begin{bmatrix} \mathbf{I}_2 & \mathbf{A}_0 & \frac{1}{2}\mathbf{A}_0^2 + \frac{1}{2}\mathbf{A}_1 \end{bmatrix}. \tag{69}$$

First five truncation approximants constructed for regular (minimal excessive) space extension and the exact solution of the system are compared through the variation of their Frobenius norms in time over the interval [0, 1] in Fig. 3. The plotting fashion is

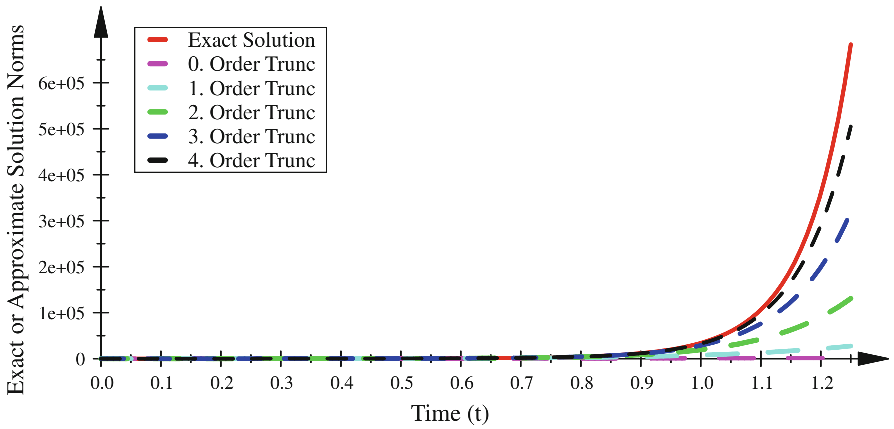

Fig. 4 The comparison of the Frobenius norms of the exact solution and first five truncation approximants constructed via a higher than regular space extension ($m_{ex} = 3$) for the system given through (64)

exactly same as before. These curves evidently show the positive effect of the excessive space extension on the quality of the same type approximants. The dimension of the extended space has been one increased and this has resulted in an extension of the t interval to produce the same quality of approximation. Now in this case the same quality of the second ex ample for $t \in [0, 0.7]$ is obtained in a wider interval $t \in [0, 1]$.

Our last example is again for the system given by (64). However this time we use a higher than the regular excessive space extension. m_{ex} is taken 3. The plots are constructed in the same fashion of the previous cases. However the interval is now [0, 1.25] (Fig. 4).

7 Concluding remarks and further readings

We have developed a new method to solve linear matrix ordinary differential equations (MODE) via space extension. We have focused on the case where the equation is homogeneous and the unknown matrix function is premultiplied by a given matrix valued function. For standardization without any loss of generality we have assumed that the unknown matrix function of the problem becomes unit matrix at the beginning of the time, or in other words, its initial value is given as unit matrix. The left factor of the matrix unknown is assumed to be a polynomial. Since this coefficient matrix is analytic in every finite region of the independent variable's complex plane, the solution of the MODE under consideration remains in all finite regions of same type due to the theory of ordinary differential equations [3].

We have increased the number of the unknown matrices by partitioning the solution of the MODE such that each new unknown depends on the independent variable via its an appropriately chosen power which is defined as "Space Extension Order". This has resulted in a new but somehow universalized MODE whose series solution provides us a two term recursion. The universal MODE has polar type singularities at $t = 0$ with powers from -1 to $-m$, where m is one less than the space extension order.

Although it might have seemed to be trivial, we have proven the convergence for all finite independent variable values and constructed error estimates for approximants constructed by truncating the series solution at a prescribed power from the lowest power.

As it is mentioned before, the explicit goal of this work is to construct a new method in order to solve first order linear ordinary differential equations efficiently. Differential equations are one of the most important phenomena which is used to model the nature. There are, of course, substantial techniques in order to solve differential equations [3,13–16]. Although many methods have been developed in order to solve systems of ordinary differential equations, the cases where the number of unknown functions are high, may cause numerical and computational difficulties. The importance of this paper arises from the fact that a universal form is obtained by using space extension concept and solution to the system of differential equations is obtained via this common form which yields the solution to a two term recursion formulae. A detailed discussion of space extension concept can be found in [7]. The recent researches and discussions on space extension approach exist in [8,17,18]. These works provide some preliminary applications of space extension concept on different types of ordinary differential equations and partial differential equations.

The space extension order can be chosen equal the number of the powers in the coefficient polynomial matrix. This is the lowest order possibility and we have called it “Minimal Space Extension”. Any order exceeding the order of the minimal space extension has been called “Regular Space Extension” since it uses a minimal block structure in the coefficient matrices (minimal space extension excludes one part of the block diagonal structure, highest power coefficient of the given polynomial matrix does not appear there). All higher order excessive space extensions have been called “Excessive Space Extension” and we have shown via illustrative examples that the approximation quality of a chosen truncation approximant increases as the order of the excessive space extension grows. This also means that the range of the quality for a specific truncation approximant broadens as the excessive space extension increases.

Application of space extension concept to first-order systems of ordinary differential equations with matrix coefficients converts the equation into a special universal form which is called as “Okubo Form” introduced to literature by Okubo [19–21]. In our previous works [8,17,18], we converted the equations into Okubo Form and tried to obtain series solution via this form. However, the series solution of Okubo Form exposes branch points which contain finite Reimann sheets. In this work, we tried to avoid these brunch points by making some modifications. This new structure also enabled us to obtain a two term recursion formula for the solution of differential equations.

The method presented here, in fact is based on somehow a series coefficient reordering and regrouping. In this logic, the computational complexity may be considered not positively affected except some matrix algebraic facilitations which may also be important in various theoretical issues like convergence, error estimates, and global behavior analyses. This method here can be empowered by using some expansions at the other end point of the interval under consideration [22]. One way is to use perturbation expansions, however it will be the subject of our a close future paper. The limitations by the polynomiality of the coefficient matrix is not a great restriction.

In fact it is quite helpful in the case where the nonpolynomial coefficient matrix of the considered problem can be approximated by polynomials. The cases of MODEs where the unknown matrix appears to be multiplied by given matrices, polynomial or nonpolynomial, from both left and right has been investigated by the authors in the space extension perspective as a beginning step. Now, after this work, what we had done there can be adapted to the case of excessive space extension application as well. This may be the topic for the one of the authors' future papers.

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